# The motivic real Milnor fibres

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# THE MOTIVIC REAL MILNOR FIBRES

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Abstract. Given a polynomial with real coefficients, we produce a motivic analog of a simple identity that relates the complex conjugation and the monodromy of the Milnor fibre of its complexification. To that purpose, we introduce motivic Zeta functions that take into account complex conjugation and monodromy.

## 1. Introduction

The monodromy homeomorphism on the Milnor fibre of a complex polynomial is a wonderful invariant saying much about the singularity of the polynomial (cf [7] for instance for a survey about monodromy). Equipped with the monodromy operator, this topological object is closely related to the algebraic nature of the polynomial.

When the polynomial has real coefficients, one may consider a positive and a negative Milnor fibres corresponding to the sign of the given polynomial close to the singularity. The relations between those positive and negative Milnor fibres, the Milnor fibre of the complexification and the monodromy operator have been investigated by several authors, notably S. Gusein-Zade [9], A. Dimca and L. Paunescu [6], or studied from a topological point of view by N. A'Campo [1]. In this direction C. McCrory and A. Parusiński have procuded in [12] a geometric monodromy homeomorphism $h$ on the Milnor fibre of the complexification of the real polynomial whose satisfies together with complex conjugation $c$ a simple identity $chch = 1$. This identity has nice consequences on the algebraic monodromy and the computations of Euler characteristics of real Milnor fibres.

More precisely, let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a real polynomial function, and consider $f_{\mathbb{C}} : \mathbb{C}^d \longrightarrow \mathbb{C}$ its complexification. The Milnor fibration of $f_{\mathbb{C}}$ at the origin is the restriction of $f_{\mathbb{C}}$ to a sufficiently small neigborhood of the origin in $\mathbb{C}^d$ of the inverse image by $f_{\mathbb{C}}$ of a sufficiently small circle around the origin in $\mathbb{C}$. Choose as the Milnor fibre $F$ the fibre of $f_{\mathbb{C}}$ over the positive real number in the circle. Denote by $c$ the complex conjugation acting on $F$. Then the algebraic monodromy $h : H_*(F, \mathbb{C}) \longrightarrow H_*(F, \mathbb{C})$ and the complex conjugation satisfy also

*Date*: February 25, 2011.

1991 *Mathematics Subject Classification.* 14B05, 14P20, 14P25, 32S15.

The author has been supported by the ANR project ANR-08-JCJC-0118-01.

the relation $chch = 1$. as a consequence [12] the the complex conjugation exchanges the eigenspaces for the action of the monodromy corresponding to complex conjugate eigenvalues: $cH_*(F, \mathbb{C})_\lambda = H_*(F, \mathbb{C})_{\overline{\lambda}}$ where $H_*(F, \mathbb{C})_\lambda = \ker(h - \lambda I)^N$, for $N$ large enough, is the generalized eigenspace of $\lambda$.

Denote by $F_+$ the fixed part of the Milnor fibre $F$ under $c$, called the positive real Milnor fibre. The Euler characteristic of $F_+$ is given by the Lefschetz number of $c$. However,as $c$ interchanges $H_\lambda$ and $H_{\overline{\lambda}}$ the trace of $c$ on $H_\lambda \oplus H_{\overline{\lambda}}$ vanishes. In particular only the eigenvalues 1 and $-1$ play a role in the computation of $\chi(F_+)$ [12]:

$$\chi(F_+) = \sum_{\epsilon=\pm 1} \sum_i (-1)^i Tr[c : H_i(F, \mathbb{C})_\epsilon \longrightarrow H_i(F, \mathbb{C})_\epsilon].$$

The real negative Milnor $F_-$ fibre is the fixed part under the complex conjugation of the Milnor fibre over the negative real number in the circle. It corresponds also to the fixed part of $F$ under the complex conjugation given by $ch$, and a similar formula holds for $\chi(F_-)$.

J. Denef and F. Loeser have introduced a motivic Milnor fibre that coincides with the classical Milnor fibre with its monodromy operator at the level of Hodge structures [4].

In this paper, we are interested in a motivic analog the relation between monodromy and complex conjugation for a real polynomial. With complex polynomial, J. Denef and F. Loeser have introduced a motivic Milnor fibre defined via the monodromic zeta function using motivic integration [5]. Their motivic Milnor fibre, which lives in a monodromic Grothendieck ring, coincides with the classical Milnor fibre with its monodromy operator at the level of Hodge structures [4].

Inspired by the complex context, we introduce motivic positive and negative Milnor fibres for a real polynomial, considering the Zeta function of the complexified polynomial with action of the complex conjugation. These Zeta functions are in particular different from that considered in [11, 8] which only deals with the real aspect. We produce then a motivic analog of the relation $chch = 1$ between complex conjugation and monodromy at the level of the motivic real Milnor fibres. We show in particular that such a simple relation already exists at the level of the spaces of arcs associated with a polynomial with real coefficients. This remark enables to define motivic real Milnor fibre as objects in a Grothendieck group of algebraic varieties over $\mathbb{R}$ endowed with an action of the pro-finite group $\hat{\mu}$ on its complexification, compatible with the real structure. We introduce finally a Grothendieck group of Hodge structure over $\mathbb{R}$ in order to state the correspondence between the motivic real Milnor and the classical Milnor fibres, in complete analogy with the complex case [4].

**Acknowledgements.** The author wish to thank Michel Merle and Johannes Nicaise for motivating discussions on the subject.

## 2. Grothendieck ring of varieties over $\mathbb{R}$

By an algebraic variety over a field $k$ we mean a reduced separated scheme of finite type over $k$.

2.1. **Complex case.** The Grothendieck group $K_0(Var_{\mathbb{C}})$ of complex algebraic varieties is the quotient of the free abelian generated by isomorphism classes $[X]$ of algebraic varieties over over $\mathbb{C}$ by the subgroup generated by the relations

$$[X] - [Y] - [X \setminus Y]$$

for $Y \subset X$ a closed subvariety. The product of varieties induces the ring structure on $K_0(Var_{\mathbb{C}})$.

We consider similarly the Grothendieck group $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ of algebraic varieties $X$ over $\mathbb{C}$ endowed with a good $\hat{\mu}$-action, where $\hat{\mu}$ denotes the pro-cyclic group of roots of unity (that is the action factorises through an action by a group of $n$-th roots of unity). Namely $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ is the free abelian group generated by couple $(X, \gamma)$ with $\gamma$ an automorphism of finite order, submitted to the relations

- $[X, \gamma] = [X', \gamma']$ if there exists an $\hat{\mu}$-equivariant isomorphism between $X$ and $X'$,
- $[X, \gamma] = [Y, \gamma] + [X \setminus Y, \gamma]$ if $Y \subset X$ is a closed subvariety over $\mathbb{C}$ endowed with the induced action of $\hat{\mu}$,
- $[X \times V, \gamma] = [X \times \mathbb{A}^n, \gamma]$ where $V$ is a $n$-dimensional affine space with any linear action, and $\mathbb{A}^n$ is endowed with the trivial action.

We endorse the Grothendieck group $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ with a ring structure by considering the diagonal action of the product.

**Remark 2.1.** A simple realization of $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ is the group morphism into $\mathbb{Z}$ given by

$$X \mapsto \sum_i (-1)^i \dim H^i(X, \mathbb{C})_\lambda,$$

where $H^*(X, \mathbb{C})_\lambda$ is the part of $H^*(X, \mathbb{C})$ on which $\hat{\mu}$ acts by multiplication by $\lambda$ [4].

2.2. **Varieties over $\mathbb{R}$.** To a variety $X$ over $\mathbb{R}$, we can associate its complexification $X \otimes_{\mathbb{R}} \mathbb{C}$ which is a variety over $\mathbb{C}$ endowed with an antiholomorphic involution coming from the complex conjugation over $\mathbb{C}$. Conversely, to a quasi-projective variety $Y$ over $\mathbb{C}$ endowed with an antiholomorphic involution can be associated a unique variety $X$ over $\mathbb{R}$ such that $Y \simeq X \otimes_{\mathbb{R}} \mathbb{C}$ ([10], exercise 4.7 p106).

Remark that a given algebraic variety over $\mathbb{C}$ may be given different real structures. For example, consider the affine line $\mathbb{A}^1_{\mathbb{R}}$, and endowed its complexification with an action of the group $\mu_n$ of $n$-roots of unity by multiplication. Then, for $\zeta \in \mu_n$, the composition $ch$ of the complex conjugation on $\mathbb{A}^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{A}^1_{\mathbb{C}}$ with the multiplication $h$ by $\zeta$ is an antiholomorphic involution, which gives to $\mathbb{A}^1_{\mathbb{C}}$ another real structure.

The Grothendieck rings $K_0(\mathrm{Var}_{\mathbb{R}})$ of algebraic varieties over $\mathbb{R}$ is defined similarly to the complex case: as a group it is freely generated by the isomorphism classes of algebraic varieties over $\mathbb{R}$, subject to the relations

$$[X] - [Y] - [X \setminus Y]$$

for $Y \subset X$ a closed subvariety. The product of varieties induces the ring structure on $K_0(Var_{\mathbb{R}})$. By extension of scalars we obtain ring morphisms from $K_0(\mathrm{Var}_{\mathbb{R}})$ to $K_0(\mathrm{Var}_{\mathbb{C}})$.

**Remark 2.2.** If we restrict ourself to quasi-projective varieties, taking the fixed point of the complex points $X(\mathbb{C})$ of an algebraic variety $X$ over $\mathbb{R}$ under the complex conjugation gives a real algebraic variety in the sense of [2], and we obtain that way a ring morphism from the Grothendieck rings $K_0(\mathrm{Var}_{\mathbb{R}})$ of algebraic varieties over $\mathbb{R}$ to the Grothendieck rings $K_0(\mathrm{Var}_{\mathbb{R}})$ of real algebraic varieties considered already in [13]. In particular we may consider a realization of $K_0(\mathrm{Var}_{\mathbb{R}})$ into the real polynomial ring in one variable via the virtual Poincaré polynomial [13].

2.3. **Equivariant case.** We define similarly the Grothendieck rings $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{R}})$ of algebraic varieties over $\mathbb{R}$ endowed with a good $\hat{\mu}$-action over $X \otimes_{\mathbb{R}} \mathbb{C}$ such that if $\sigma$ denotes the complex conjugation and $\gamma$ the finite order action then $\sigma\gamma\sigma\gamma = 1$.

By extension of scalars we obtain ring morphisms from $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{R}})$ to $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$.

For such an $X$, the complex conjugation $\sigma$ exchanges the characteristic spaces of $H^*(X(\mathbb{C}), \mathbb{C})$ associated to complex conjugate eigenvalues for $\gamma$ ([12]). Namely, if $H^*(X(\mathbb{C}), \mathbb{C})_\lambda$ denotes the eigenspace associated to the eigenvalue $\lambda$ for the action of $\gamma$ on $H^*(X(\mathbb{C}), \mathbb{C})$, then $\sigma H^*(X(\mathbb{C}), \mathbb{C})_\lambda = H^*(X(\mathbb{C}), \mathbb{C})_{\overline{\lambda}}$. Actually $\gamma\sigma(\gamma - \lambda Id)\sigma = \lambda(\lambda^{-1}I - \gamma)$ and $\lambda^{-1} = \overline{\lambda}$ because $\gamma$ has finite order.

In particular the complex conjugation $\sigma$ stabilizes the eigenspace $H^*(X(\mathbb{C}), \mathbb{C})_\alpha$ of $\gamma$ corresponding to a real eigenvalue $\alpha \in \{\pm 1\}$. Denote by $H^*(X(\mathbb{C}), \mathbb{C})_{\alpha,\beta}$ the characteristic space corresponding the eigenvalue $\beta \in \{\pm 1\}$ for the action of $\sigma$ on $H^*(X(\mathbb{C}), \mathbb{C})_\alpha$. We define

$$\chi_{\alpha,\beta} = \sum_i (-1)^i \dim H^i(X(\mathbb{C}), \mathbb{C})_{\alpha,\beta}.$$

**Lemma 2.3.** *The map $\chi_{\alpha,\beta} : K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{R}}) \longrightarrow \mathbb{Z}$ is a group morphism.*

*Proof.* Let $X$ be algebraic variety over $\mathbb{R}$, endowed with an automorphism $\gamma$ of finite order such that $\sigma\gamma\sigma\gamma = 1$, where $\sigma$ denotes the complex conjugation of $X(\mathbb{C})$. For a closed inclusion $Y \subset X$ of real algebraic varieties such that $\gamma$ stabilises $Y$, we have a long exact sequence

$$\to H^i(X \setminus Y) \to H^i(X) \to H^i(Y) \to H^{i+1}(X \setminus Y) \to$$

that induces a long exact sequence for the characteristics spaces associated to a real eigenvalue $\alpha \in \{\pm 1\}$ for the action induced by $\gamma$

$$\to H^i(X \setminus Y)_\alpha \to H^i(X)_\alpha \to H^i(Y)_\alpha \to H^{i+1}(X \setminus Y)_\alpha \to$$

As $\alpha$ is real $\sigma$ acts on these eigenspaces, and therefore the sequence induces in the same way an exact sequence for the characteristics spaces associated to an eigenvalue for the action of $\sigma$:

$$\to H^i(X \setminus Y)_{\alpha,\beta} \to H^i(X)_{\alpha,\beta} \to H^i(Y)_{\alpha,\beta} \to H^{i+1}(X \setminus Y)_{\alpha,\beta} \to$$

□

**Remark 2.4.** For the complex Milnor fibre $F$ endowed with complex conjugation $c$, the Lefschetz number of the complex conjugation $c$ can be expressed as

$$\chi(F_+) = \chi_{1,1}(F) + \chi_{1,-1}(F) + \chi_{-1,1}(F) + \chi_{-1,-1}(F).$$

## 3. Motivic Zeta functions over $\mathbb{R}$

In this section we introduce motivic Zeta functions for a real algebraic polynomial germ, in analogy with the monodromic motivic Zeta function in [4]. We refer to [5] for the general theory of motivic integration, and just recal that for an algebraic variety $X$ over $\mathbb{C}$, the complex points of the arc space $\mathcal{L}(X)$ are the $\mathbb{C}[[t]]$-rational points on $X$, whereas the complex points of the truncated arc space $\mathcal{L}_n(X)$ are the $\mathbb{C}[[t]]/t^{n+1}$-rational points on $X$.

3.1. **Arcs spaces with conjugation.** For $X$ an algebraic variety over $\mathbb{R}$ of dimension $d$, we will consider two antiholomorphic involutions on the spaces of arcs $\mathcal{L}_n(X_\mathbb{C})$ of its complexification $X_\mathbb{C}$. Denote by $\gamma_n$ the natural action of the group $\mu_n$ of $n$-th roots of unity on $\mathcal{L}_n(X_\mathbb{C})$ given by $\gamma_n(\phi(t)) = \phi(e^{2\pi i/n}t)$. We defined $c_n^+$ on $\mathcal{L}_n(X_\mathbb{C})$ to be the complex conjugation $c_n^+(\phi(t)) = \overline{\phi}(t)$ on the arcs $\phi(t)$, namely if $\phi(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n$ with $a_1, \dots, a_n \in \mathbb{C}$, then

$$c_n^+(\phi(t)) = \overline{\phi}(t) = \overline{a}_1 t + \overline{a}_2 t^2 + \cdots + \overline{a}_n t^n.$$

We remark in the following lemma that the complex conjugation $c_n^+$ and the action of roots of unity satisfy a similar relation than complex conjugation and monodromy as in [9, 6, 1, 12].

**Lemma 3.1.** *The relation $c_n^+ \gamma_n c_n^+ \gamma_n = 1$ holds in $\mathcal{L}_n(X_\mathbb{C})$.*

*Proof.* Let $\phi(t) = a_1t + a_2t^2 + \cdots + a_nt^n$ be in $\mathcal{L}_n(X_\mathbb{C})$. Put $\xi = e^{2\pi i/n}$. Then

$$\begin{aligned} c_n^+\gamma_n(\phi(t)) &= c_n^+(a_1\xi t + a_2(\xi t)^2 + \cdots + a_n(\xi t)^n) \\ &= \overline{a}_1\overline{\xi}t + \overline{a}_2(\overline{\xi}t)^2 + \cdots + \overline{a}_n(\overline{\xi}t)^n \end{aligned}$$

therefore

$$\gamma_n c_n^+\gamma_n(\phi(t)) = \overline{a}_1\overline{\xi}\xi t + \overline{a}_2(\overline{\xi}\xi t)^2 + \cdots + \overline{a}_n(\overline{\xi}\xi t)^n$$

is equal to $c_n^+(\phi(t))$ since $\overline{\xi}\xi = 1$. □

We define $c_n^-$ to be $c^- = c_n^+\gamma_n$. By lemma 3.1, $c_n^-$ is again an antiholomorphic involution on $\mathcal{L}_n(X_\mathbb{C})$ that plays the role of a complex conjugation. We will see below that $c_n^-$ corresponds to the complex conjugation acting on the negative Milnor fibre.

For a dominant morphism $f : X \longrightarrow \mathbb{A}^1_\mathbb{R}$ and $n \geq 1$, we define its spaces of positive truncated arcs $\mathcal{X}_n^1(f)$ by

$$\mathcal{X}_n^1(f) = \{\phi \in \mathcal{L}_n(X) | f(\phi(t)) = t^n \mod t^{n+1}\}.$$

We define similarly its spaces of negative truncated arcs $\mathcal{X}_n^{-1}(f)$ by

$$\mathcal{X}_n^{-1}(f) = \{\phi \in \mathcal{L}_n(X) | f(\phi(t)) = -t^n \mod t^{n+1}\}.$$

Similar arc spaces have alreay been studied in the context of real polynomials in [11, 8], without taking into account the role of complex conjugation.

We have a natural action of $\mu_n$ on the complexification $\mathcal{X}_n^{\pm 1}(f_\mathbb{C})$ of $\mathcal{X}_n^{\pm 1}(f)$ given by multiplication of the indeterminate $t$ by $n$-th roots of unity. Note that the complex conjugations $c_n^+$ and $c_n^-$ act also on $\mathcal{X}_n^{\pm 1}(f_\mathbb{C})$ by restriction. For $c_n^+$, the set of fixed points of $\mathcal{X}_n^1(f_\mathbb{C})$ is the set of truncated arcs with real coefficients that realizes the order $n$ after composition with $f$, with a leading coefficient equals to 1. For $c_n^-$, the set of fixed points is related to those arcs whose leading coefficient is equal to $-1$.

**Lemma 3.2.** *Let $\tau = e^{\pi i/n}$. The map $\phi \mapsto \phi(\tau t)$ realizes an isomorphism between the complex algebraic variety $\mathcal{X}_n^1(f_\mathbb{C})$ endowed with the real structure given by $c_n^-$, and the complex algebraic variety $\mathcal{X}_n^{-1}(f_\mathbb{C})$ endowed with the real structure given by $c_n^+$.*

*Proof.* Let $\phi \in \mathcal{X}_n^1(f)$ such that $c_n^-(\phi) = \phi$. If $\phi(t) = a_1t + \cdots + a_nt^n$, then for $i \in \{1, \ldots, n\}$ we obtain $\overline{a_i\xi^i} = a_i$, or equivalently $\overline{a_i\tau^i} = a_i\tau^i$. In particular $\phi(t)$ is defined over $\mathbb{R}$ with the real structure defined by $c_n^-$ if and only if $\phi(\tau t)$ is defined over $\mathbb{R}$ with the real structure defined by $c_n^+$.

Moreover

$$f \circ \phi(\tau t) = (\tau t)^n \mod t^{n+1} = -t^n \mod t^{n+1}$$

therefore $\phi \mapsto \phi(\tau t)$ maps $\mathcal{X}_n^1(f_\mathbb{C})$ to $\mathcal{X}_n^{-1}(f_\mathbb{C})$. □

In particular, the complex conjugations $c_n^+$ and $c_n^-$ considered on $\mathcal{X}_n^1(f_{\mathbb{C}})$ enable to study on the same arc space the question of signs for $f$. We will consider the classes $[\mathcal{X}_n^1(f_{\mathbb{C}}), \mu_n, c^+]$ and $[\mathcal{X}_n^1(f_{\mathbb{C}}), \mu_n, c^-]$ in $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{R}})$ of $\mathcal{X}_n^1(f_{\mathbb{C}})$ endowed with the classical $\mu_n$-action and the antiholomorphic actions induced by $c_n^+$ and $c_n^-$ respectively.

We define then the positive real motivic zeta function $Z_f^+(T)$ of $f : X \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ as the power series over $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{R}})[\mathbb{L}^{-1}]$ obtained by collecting the measure of the complex truncated arcs endowed with the positive complex conjugation $c_n^+$:

$$Z_f^+(T) := \sum_{n \geq 1} [\mathcal{X}_n^1(f_{\mathbb{C}}), \mu_n, c_n^+] \mathbb{L}^{-nd} T^n.$$

We define similarly the negative real motivic zeta function $Z_f^-(T)$ of $f$ using the negative complex conjugation

$$Z_f^-(T) := \sum_{n \geq 1} [\mathcal{X}_n^1(f_{\mathbb{C}}), \mu_n, c_n^-] \mathbb{L}^{-nd} T^n.$$

3.2. **Denef & Loeser formula.** Let $(Y, h)$ be a real resolution of $f : X \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ in the sense that is $Y$ is a nonsingular irreducible algebraic variety over $\mathbb{R}$, $h$ is proper, the restriction of $h$ to $Y \setminus h^{-1}(X_0)$ is an isomorphism onto $X \setminus X_0$, and $h^{-1}(X_0)$ has only normal crossings, where $X_0$ denotes the zero set of $f$ in $X$.

We denote by $h^{-1}(X_0) = \cup_{j \in J} E_j$ the decomposition into irreducible components of $h^{-1}(X_0)$, by $N_j$ the multiplicity of $f$ along $E_j$, and by $\nu_j - 1$ the multiplicity of the Jacobian determinant of $h$ along $E_j$. For a subset $I \subset J$, we consider $E_I^o = \cap_{i \in I} E_i \setminus \cup_{j \in J \setminus I} E_j$.

Let $m_I = gcd(N_i)_{i \in I}$. In order to express the zeta functions in terms of the resolution of $f$, we define a positive Galois cover $\widetilde{E_I^{o,+}}$ of $E_I^o$ with Galois group $\mu_{m_I}$, together with a negative Galois cover $\widetilde{E_I^{o,-}}$, as follows. They are defined on an affine chart $U$ where $f \circ h = uv^{m_I}$ with $u$ a unit on $U$ and $v$ a morphism from $U$ to $A^1_{\mathbb{C}}$, both equivariant with respect to the complex conjugation since $h$ is defined over $\mathbb{R}$, by

$$\widetilde{E_I^{o,+}} \cap U := \{(z, x) \in A^1_{\mathbb{C}} \times (E_I^o \cap U) | z^{m_I} = +u^{-1}(x)\}$$

in the positive case and by

$$\widetilde{E_I^{o,-}} \cap U := \{(z, x) \in A^1_{\mathbb{C}} \times (E_I^o \cap U) | z^{m_I} = -u^{-1}(x)\}$$

in the negative case. We obtain the Galois cover $\widetilde{E_I^{o,\pm}}$ over $E_I^o$ by gluing together the covers $\widetilde{E_I^{o,\pm}} \cap U$ over $E_I^o \cap U$. It is naturally endowed with a $\mu_{m_I}$-action induced by $\gamma : (z, x) \mapsto (e^{2\pi i/m_I} z, x)$. Moreover we can endowed it with two antiholomorphic involutions given by $\sigma^+(z, x) = (\overline{z}, \overline{x})$ and $\sigma^-(z, x) = \sigma^+ \gamma(z, x) = (e^{-2\pi i/m_I} \overline{z}, \overline{x})$.

**Proposition 3.3.** *In* $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{R}})$

$$Z_f^{\pm}(T) = \sum_{I \neq \emptyset} (\mathbb{L}-1)^{|I|-1} [\widetilde{E_I^{o,\pm}}, \hat{\mu}, c^{\pm}] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}}.$$

*Proof.* The proof in the complex case carries over litteraly in this setting (cf [4]). □

This formula makes sense to evaluate $Z_f^{\pm}(T)$ as $T$ go to $\infty$. By analogy with the complex setting, we define the positive motivic Milnor fibre $\mathcal{S}^+$ and the negative motivic Milnor fibre $\mathcal{S}^-$ to be the elements of $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{R}})$ defined by

$$\mathcal{S}^{\pm} = - \lim_{T \to \infty} Z_f^{\pm}(T).$$

In particular we obtain from proposition 3.3 an expression for $\mathcal{S}^{\pm}$ in terms of a resolution of $f$:

**Corollary 3.4.** *With the notations of proposition 3.3,*

$$\mathcal{S}^{\pm} = \sum_{I \neq \emptyset} (1-\mathbb{L})^{|I|-1} [\widetilde{E_I^{o,\pm}}, \hat{\mu}, c^{\pm}].$$

## 4. Hodge realization over $\mathbb{R}$

We introduce the context in order to relate the motivic positive and negative Milnor fibres with the classical Milnor fibres. Together with proposition 3.3, the key point is a formula that expresses the nearby cycles of a given polynomial in terms of a resolution of its singularities in a convenient Grothendieck ring of Hodge structures (cf [4] or [15], Corollary 11.26).

4.1. **Hodge realization in the complex case.** The comparison theorem between the motivic real Milnor fibres and the classical real Milnor fibres is simply an adaptation of the corresponding result for complex numbers [4], taking into account the complex conjugation. Therefore in this section we recall briefly the Grothendieck rings corresponding to the Hodge structure of an algebraic variety over $\mathbb{C}$, following [14].

A $\mathbb{Q}$-Hodge structure consists of a $\mathbb{Q}$-vector space $H$ with a direct decomposition

$$H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}, \quad \text{with} \quad H^{p,q} = \overline{H^{q,p}}$$

of its complexification. The Grothendieck group $K_0(HS)$ of $\mathbb{Q}$-Hodge structure is defined to be the free abelian group on the isomorphism classes of $\mathbb{Q}$-Hodge structure modulo the relations $[H] = [H'] + [H'']$ for an sequence

$$0 \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow 0$$

of $\mathbb{Q}$-Hodge structures. The tensor product gives a product structure on $K_0(HS)$. The dimensions $h^{p,q}(H) = \dim_{\mathbb{R}} H^{p,q}$ are the Hodge numbers of the Hodge structure. Collecting the Hodge numbers in the Hodge polynomial $E(H) = \sum_{p,q} h^{p,q}(H) u^p v^q$, gives a ring morphism

$$E : K_0(HS) \longrightarrow \mathbb{Z}[u, v, u^{-1}, v^{-1}].$$

In case there exists an integer $k$ such that $h^{p,q} = 0$ when $p + q \neq k$, we say that the Hodge structure is pure of weight $k$. An example of a pure Hodge structure of weight $k$ is given by the $k$-cohomology of a proper regular algebraic variety over $\mathbb{C}$. More generally, every complex algebraic variety admits a mixed Hodge structure on its cohomology with compact supports ([3]), that we can see as a direct sum of pure Hodge structure. This leads to a ring morphism

$$\chi_H : K_0(Var_{\mathbb{C}}) \longrightarrow K_0(HS)$$

defined by $\chi_H(X) = \sum_i (-1)^i [H^i_c(X, \mathbb{Q})]$ and the Hodge polynomial induces a ring morphism

$$e : K_0(Var) \longrightarrow \mathbb{Z}[u, v, u^{-1}, v^{-1}].$$

We will consider moreover Hodge structure endowed with an endomorphism with finite order. This leads to the Grothendieck ring $K_0^{\hat{\mu}}(HS)$ of Hodge structure with an action of $\hat{\mu}$. The Hodge characteristics $\chi_H$ extends to the equivariant setting:

$$\chi_H : K_0^{\hat{\mu}}(Var_{\mathbb{C}}) \longrightarrow K_0^{\hat{\mu}}(HS).$$

Note however that the Hodge polynomial only extends to a group morphism.

4.2. **Hodge realization over $\mathbb{R}$.** A Hodge structure over $\mathbb{R}$ is the data of a Hodge structure $H$ with an $\mathbb{Q}$-linear involution $\sigma$ such that $\sigma(H^{p,q}) = \overline{H^{q,p}}$. The Grothendieck ring $K_0(HS_{\mathbb{R}})$ of Hodge structure over $\mathbb{R}$ is then defined similarly to the complex case. As an example, the Hodge structure on the cohomology of the complexification $X(\mathbb{C})$ of a regular algebraic variety $X$ over $\mathbb{R}$ is equipped with a real structure [16] (oand even non regular by [3] since the weight filtration is preserved by the complex conjugation), the $\mathbb{Q}$-linear involution $\sigma$ being induced by the complex conjugation on $X(\mathbb{C})$. More precisely, denote by $\sigma$ again the involution on $H^*(X(\mathbb{C}), \mathbb{Q})$ induce by the complex conjugation on $X(\mathbb{C})$, and extend it to $H^*(X(\mathbb{C}), \mathbb{C})$. Denote by $H^{p,q}(X(\mathbb{C}))$ the subspaces given by the Hodge decomposition of $H^*(X(\mathbb{C}), \mathbb{C})$. Then $\sigma H^{p,q}(X(\mathbb{C})) = H^{q,p}(X(\mathbb{C}))$.

**Remark 4.1.** As a consequence the eigenspaces for the values 1 and $-1$ of the action of $\sigma$ on $H^{p,q}(X(\mathbb{C})) \oplus H^{q,p}(X(\mathbb{C}))$ have the same dimension if $p \neq q$.

We will consider moreover Hodge structure over $\mathbb{R}$ endowed with an endomorphism with finite order compatible with the real structure, namely a Hodge structure $(H, \sigma)$ over $\mathbb{R}$ together with an automorphism of finite order $\gamma$ on $H \otimes_{\mathbb{Q}} \mathbb{C}$ such that $\sigma\gamma\sigma\gamma = 1$.

This leads to the Grothendieck ring $K_0^{\hat{\mu}}(HS_{\mathbb{R}})$ of Hodge structure over $\mathbb{R}$ with an action of $\hat{\mu}$, together with the Hodge realization

$$\chi_H : K_0^{\hat{\mu}}(Var_{\mathbb{R}}) \longrightarrow K_0^{\hat{\mu}}(HS_{\mathbb{R}}).$$

4.3. **Nearby cycles and the comparison theorem.** Consider a smooth irreducible variety $X$ over $\mathbb{R}$, and a dominant morphism $f : X \longrightarrow \mathbb{A}^1_{\mathbb{R}}$. We denote by $X_{\mathbb{C}}$ the complexification $X \otimes_{\mathbb{R}} \mathbb{C}$ of $X$, and by $f_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{C} = A^1_{\mathbb{C}}$ the complexification of $f$.

We recall the definition of the nearby cycles complex of $f_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow A^1_{\mathbb{C}}$. We denote by $X_0$ the hypersurface $\{f^{-1}(0)\}$ defined by $f_{\mathbb{C}}$ in $X_{\mathbb{C}}$, by $i : X_{\mathbb{C}} \setminus X_0 \longrightarrow X_{\mathbb{C}}$ and by $j : X_0 \longrightarrow X_{\mathbb{C}}$ the induced inclusions. Restricted to $X_{\mathbb{C}} \setminus X_0$, the morphism $f_{\mathbb{C}}$ takes values in $\mathbb{G}_m(\mathbb{C}) \subset A^1_{\mathbb{C}}$. Consider the universal cover $\mathbb{C} \longrightarrow \mathbb{G}_m(\mathbb{C}) : z \mapsto e^z$, and let $X_\infty$ be the fibered product of $\mathbb{C}$ and $X_{\mathbb{C}} \setminus X_0$ over $\mathbb{G}_m(\mathbb{C})$:

$$\begin{tikzcd}
X_\infty \arrow[r, "\pi"] \arrow[d] & X_{\mathbb{C}} \setminus X_0 \arrow[r, "i"] \arrow[d] & X_{\mathbb{C}} \arrow[d, "f_{\mathbb{C}}"] & X_0 \arrow[l, "j"'] \arrow[d] \\
\mathbb{C} \arrow[r, "exp"'] & \mathbb{G}_m(\mathbb{C}) \arrow[r] & A^1_{\mathbb{C}} & 0 \arrow[l]
\end{tikzcd}$$

The nearby cycles complex of $f_{\mathbb{C}}$ is the object $\psi_{f_{\mathbb{C}}} = j^* R i_* (\pi_* \mathbb{Z}_{X_\infty})$ in the derived category $D^b(X_0)$ of sheaves on $X_0$.

At the level of the universal cover $\mathbb{C} \longrightarrow \mathbb{G}_m(\mathbb{C}) : z \mapsto e^z$, the monodromy action is given by the translation $M : z \mapsto z + 2\pi i$. Moreover the complex conjugations over $X_{\mathbb{C}}$, $A^1_{\mathbb{C}}$ and $\mathbb{C}$ are compatible in the sense that $f_{\mathbb{C}}(\overline{x}) = \overline{f_{\mathbb{C}}(x)}$ for $x \in X_{\mathbb{C}}$. Denote by $c^+$ the complex conjugation over the universal cover $\mathbb{C}$ of $\mathbb{G}_m(\mathbb{C})$. We define $c^-$, the "negative complex conjugation", to be the composite $c^+ M$. The conjugation $c^- : z \mapsto z - 2\pi i$ is an antiholomorphic involution over $\mathbb{C}$. The actions of $c^\pm$ lifts to the fibered product $X_\infty$ as the action $\sigma^\pm : (x, z) \mapsto (\overline{x}, c^\pm(z))$ since $\overline{e^z} = e^{c^\pm(z)}$. The diagram above is then equivariant with respect to the action of $c^\pm$, therefore we can extend this action to the nearby cycles complex $\psi_{f_{\mathbb{C}}}$. We denote it by $\psi_f^+$ when considered with the action of $c^+$ and by $\psi_f^-$ when considered with the action of $c^-$.

We can express the nearby cycles complex of $f$ if terms of a resolution of the singularities of $f$. The proof given in [15] carries over literally in our situation. Let $(Y, h)$ be a resolution of the singularities of $f$ as in proposition 3.3.

**Proposition 4.2.** [15] *With the notations of proposition 3.3, the Hodge character of $\psi_f^{\pm}$ can be expressed in $K_0^{\hat{\mu}}(HS_{\mathbb{R}})$ by*

$$\chi^H(\psi_f^{\pm}) = \sum_{I \neq \emptyset} (1-\mathbb{L})^{|I|-1} \chi^H(\widetilde{E_I^{o,\pm}}, \hat{\mu}, c^{\pm}).$$

**Remark 4.3.** In [15], the formula for the Hodge characteristic of the nearby cycles complex in expressed in terms of the $E_I$ rather than of the $E_I^0$. We pass from one to another using the additivity in the Grothendieck ring of Hodge structures.

The comparison theorem below asserts then that the motivic real Milnor fibres are good motivic representatives for the real Milnor fibres, with compatible action of the complex conjugation and monodromy. It offers at a motivic level an interpretation of the relations between monodromy and complex conjugation as was previously observed in [1, 6, 9, 12].

**Theorem 4.4.** *The positive (respectively negative) real motivic Milnor fibre coincide with the Milnor fiber endowed with the positive (respectively negative) complex conjugation in the Grothendieck ring $K_0^{\hat{\mu}}(HS_{\mathbb{R}})$ of real Hodge structures:*

$$\chi^H(\mathcal{S}^{\pm}) = \chi^H(\psi_f^{\pm}).$$

*Proof.* The formula given by proposition 3.3 and that of proposition 4.2 coincide in $K_0^{\hat{\mu}}(HS_{\mathbb{R}})$. □

Considering the realization via the classical Euler characteristics, we obtain as a corollary a congruence mod 2 of the positive and negative Milnor, in the same spirit as Sullivan's Theorem [17].

**Corollary 4.5.** *Let $f : X \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ be a dominant morphism. Then*

$$\chi(\psi_f^+) \equiv \chi(\psi_f^-) \mod 2.$$

*Proof.* By corollary 3.4, and with its notations, the Euler characteristic of $\psi_f^{\pm}$ computed mod 2 is equal to

$$\chi(\psi_f^{\pm}) \equiv \sum_{i \in I} \chi(\widetilde{E_i^{o,\pm}}) \mod 2.$$

However, if the multiplicity $N_i$ of $f$ along the exceptional divisor $E_i$ is odd, then $\chi(\widetilde{E_i^{o,+}}) = \chi(\widetilde{E_i^{o,-}})$. In case $N_i$ is even, then $\chi(\widetilde{E_i^{o,\pm}})$ is twice the Euler characteristic of the semi-algebraic subset of $E_i^o$ given by $\pm f \circ h > 0$, therefore

$$\chi(\widetilde{E_i^{o,\pm}}) \equiv 0 \mod 2.$$

□

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